

PREPARED FOR SUBMISSION TO JHEP

Partial resolution of complex cones over Fano \mathcal{B}

Siddharth Dwivedi and P. Ramadevi

*Department of Physics, Indian Institute of Technology Bombay,
Mumbai 400 076, India*

E-mail: siddharth@phy.iitb.ac.in, ramadevi@phy.iitb.ac.in

ABSTRACT: In our recent paper arXiv:1108.2387, we systematized inverse algorithm to obtain quiver gauge theory living on the M2-branes probing the singularities of special kind of Calabi-Yau four-folds which were complex cones over toric Fano \mathbb{P}^3 , \mathcal{B}_1 , \mathcal{B}_2 , \mathcal{B}_3 . These quiver gauge theories cannot be given a dimer tiling presentation. We use the method of partial resolution to show that the toric data of \mathbb{C}^4 , orbifolds of \mathbb{C}^4 and Fano \mathbb{P}^3 can be embedded inside the toric data of Fano \mathcal{B} . This method indirectly justifies that the two node quiver Chern-Simons theories corresponding to \mathbb{C}^4 , orbifolds of \mathbb{C}^4 and \mathbb{P}^3 can be obtained by higgsing matter fields of the three node parent quiver corresponding to Fano \mathcal{B}_i three-folds.

KEYWORDS: AdS-CFT Correspondence, M-theory

Contents

1	Introduction	1
2	Two-node quiver Chern-Simons theories	3
3	Fano \mathcal{B}_4	6
3.1	Forward algorithm	6
3.2	Inverse algorithm	9
3.3	Algebraic higgsing	10
3.4	Partial Resolution	11
4	Fano \mathcal{B}_3	12
4.1	Algebraic higgsing	14
4.2	Partial Resolution	14
4.2.1	Embedding of Fano \mathbb{P}^3 inside Fano \mathcal{B}_3	15
4.2.2	Embedding of \mathbb{C}^4 and orbifolds of \mathbb{C}^4 inside Fano \mathcal{B}_3	15
5	Fano \mathcal{B}_2	16
5.1	Partial Resolution	18
5.1.1	Embedding of Fano \mathbb{P}^3 inside Fano \mathcal{B}_2	18
6	Fano \mathcal{B}_1	19
7	Conclusions	20

1 Introduction

Initial works of Bagger-Lambert[1–3] followed by Gustavsson[4, 5], Raamsdonk[6] and Aharony-Bergman-Jafferis-Maldacena (ABJM)[7] led to a flurry of interesting papers during the last four years between supersymmetric Chern-Simons gauge theory on coincident $M2$ -branes at the tip of Calabi-Yau four folds and their string duals. In a review article[8], these developments are discussed in detail.

Martelli et al[9] discussed the gauge-gravity correspondence (AdS_4/CFT_3) for some supersymmetric Chern-Simons theories with a quiver diagram description. Earlier works of Hanany et al in the context of Calabi-Yau three-folds[10] called *forward algorithm*, can be extended to obtain Calabi-Yau four-fold toric data from $2 + 1$ dimensional quiver supersymmetric Chern-Simons theories.

An elegant combinatorial approach called dimer tilings[11, 12] which gives both the toric data and the corresponding quiver gauge theories was generalised to study quiver Chern-Simons theories[13–20]. However, the dimer tiling approach is applicable for only a class of quiver gauge theories with m -matter fields, r gauge group nodes and N_W number of terms in the superpotential W satisfying $r - m + N_W = 0$. The Chern-Simons (CS) levels of the r -nodes can be denoted by the vector $\vec{k} = (k_1, k_2, \dots, k_r)$. We observe that the toric data associated with quiver theories with CS levels \vec{k} is $GL(4, \mathbb{Z})$ related to toric data associated with the same quiver with a scaled CS levels $n\vec{k}$ where n is a non-zero integer. For a three node quiver corresponding to Fano \mathcal{B}_4 , we will present this scaling freedom of \vec{k} giving the same toric data.

One of the challenging problems was to determine quiver gauge theories corresponding to 18 toric Fano three-folds[21]. From the forward algorithm and dimer tiling method, quiver gauge theories corresponding to fourteen of the toric Fano 3-folds were determined[20]. In Ref.[22], we attempted the inverse algorithm of obtaining the quiver gauge theories for the remaining four Fano 3-folds. As expected, these quiver gauge theories does not satisfy $r - m + N_W = 0$ confirming that they cannot be given dimer tiling presentation.

The next immediate question is to understand the embeddings inside the toric Fano \mathcal{B} three-folds by the method of partial resolution. In particular, we would like to obtain the quiver Chern-Simons theory corresponding to Fano \mathbb{P}^3 by partial resolution of Fano \mathcal{B} three-folds.

Alternatively, we could determine embeddings from higgsing approach[19, 23, 24]: For the quivers with r nodes (usually called parent theories) which admit dimer tiling, we could higgs some matter fields and obtain quivers with $r - 1$ nodes (called daughter theories). Suppose we give a vacuum expectation value (VEV) to a bifundamental matter field X_{ab} where the subscript denotes that the matter field is charged $+1$ with respect to node a with Chern-Simons level k_a and charged -1 with respect to node b with Chern-Simons level k_b . This results in coalescing of two nodes a and b into a node with Chern-Simons level $k_a + k_b$. Higgsing of the three node quiver corresponding to a toric Fano \mathcal{B}_4 was discussed from dimer tilings in Ref.[19]. In fact, the daughter theories corresponds to either phase II of \mathbb{C}^4 or phase I of $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}^2$ depending upon which bifundamental field was given VEV. Imposing the scaling freedom on CS levels of the parent quiver, we can infer the toric data of the daughter theories could be \mathbb{C}^4 and orbifolds of \mathbb{C}^4 .

Another approach of higgsing called the algebraic method[23] has been applied on parent quivers[24] corresponding to some toric Fano 3-folds. We shall present the algebraic higgsing for Fano \mathcal{B}_4 and show that the toric data corresponding to daughter theories has to be \mathbb{C}^4 or orbifolds of \mathbb{C}^4 . We will also study the method of partial resolution[10] and obtain the same toric data for the daughter theories.

For the 3-node quivers corresponding to Fano $\mathcal{B}_1, \mathcal{B}_2$ and \mathcal{B}_3 , which do not admit dimer tiling presentation, we have to obtain daughter theories by the algebraic higgsing

method[24] or the partial resolution method[10]. We shall show that the algebraic higgsing on these 3-node quivers gives a trivial result and hence we have to rely only on the method of partial resolution to determine the daughter theories. Interestingly this method shows that the toric data of Fano \mathbb{P}^3 is embedded in the toric data of Fano \mathcal{B}_2 and Fano \mathcal{B}_3 .

It is appropriate to point out that we had deduced[22] the 2-node quiver diagram corresponding to complex cone over Fano \mathbb{P}^3 and proposed the CS levels by comparing the CS levels of orbifolds of \mathbb{C}^4 . Now we can use the scaling freedom in the CS levels of the parent quivers corresponding to Fano \mathcal{B}_2 and \mathcal{B}_3 to determine the CS levels of the two-node daughter quiver corresponding to Fano \mathbb{P}^3 .

The plan of the paper is as follows: In section 2, we briefly review the well known 2-node quiver Chern-Simons theory with four matter fields and their toric data. In section 3, we will first show, using forward algorithm, that the scaling of CS levels $\vec{k} \rightarrow n\vec{k}$ in the 3-node quiver corresponding to toric Fano \mathcal{B}_4 does not result in a new toric data. We also indicate that the inverse algorithm, from the toric data \mathcal{B}_4 , allows a scaling freedom in the CS level assignments of the 3-node quiver. Then, we will perform the higgsing of toric Fano \mathcal{B}_4 using the algebraic method and determine the daughter quiver theories with 2-nodes. Finally, we show that the partial resolution method gives the same daughter quivers. In section 4, we will briefly present the necessary data of the quiver corresponding to Fano \mathcal{B}_3 . Then we study the algebraic higgsing and the method of partial resolution for Fano \mathcal{B}_3 . Particularly, we show that the algebraic higgsing gives trivial result whereas the method of partial resolution gives non-trivial embeddings inside Fano \mathcal{B}_3 . In section 5, we study the method of partial resolution for Fano \mathcal{B}_2 . We present the results of partial resolution of Fano \mathcal{B}_1 in section 6. We summarize and discuss some open problems in section 7.

2 Two-node quiver Chern-Simons theories

Our aim is to study partial resolution of toric data corresponding to 3-node parent quiver resulting in a toric data corresponding to a 2-node daughter quiver. So, in this section we will briefly recapitulate the 2-node quivers with four matter fields corresponding to toric data of \mathbb{C}^4 , orbifolds of \mathbb{C}^4 and Fano \mathbb{P}^3 . There can be three possible quivers:

1. Theory with 4 bi-fundamental matter fields X_{12}^i and X_{21}^i (where $i = 1, 2$), with CS levels $(k, -k)$ and the superpotential $W = \text{Tr}[\epsilon_{ij} X_{12}^1 X_{21}^i X_{12}^2 X_{21}^j]$. For the abelian groups, $W = 0$. The quiver diagram is shown in figure 1. This theory admits tiling and the corresponding toric data is[18]:

$$\mathcal{G}_a(k) = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ 1 & 1 & 1 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & k & 0 \end{pmatrix}. \quad (2.1)$$

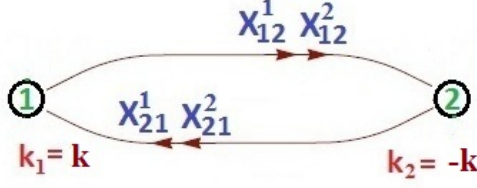


Figure 1. Quiver diagram (a)

The charge matrix Q_a for this theory given by $Q_a \cdot \mathcal{G}_a^T(k) = 0$ is trivial, $Q_a = 0$. This theory is \mathbb{Z}_k orbifold of \mathbb{C}^4 , denoted as $\mathbb{C}^4/\mathbb{Z}_k$. For $k = 1$, i.e, when the CS-levels of the theory is $(1, -1)$, there is no orbifolding action and this theory in the literature is known as Phase-I of \mathbb{C}^4 [18].

2. Theory with 2 adjoints ϕ_2^1, ϕ_2^2 present at the same node (say node 2) and two bifundamentals X_{12}, X_{21} , with CS-levels $(k, -k)$. Abelian W of this theory is again zero with non-abelian superpotential $W = \text{Tr}[X_{12}[\phi_2^1, \phi_2^2]X_{21}]$. The quiver diagram is shown in figure 2. This theory also admits tiling and the toric data for this theory is[18]:

$$\mathcal{G}_b(k) = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & k & 0 \end{pmatrix}. \quad (2.2)$$

The charge matrix Q_b of this theory given by $Q_b \cdot \mathcal{G}_b^T(k) = 0$ is also trivial, $Q_b = 0$. This is $(\mathbb{C}^2/\mathbb{Z}_k) \times \mathbb{C}^2$ theory. For $k = 1$, this theory is known as Phase-II of \mathbb{C}^4 .

It is pertinent to spell out the following obvious facts:

- (i) For non-trivial orbifolding ($k \neq 1$), we can show that the toric data $\mathcal{G}_a(k)$ is $GL(4, \mathbb{Z})$ related to $\mathcal{G}_a(nk)$, where n is a non-zero integer. So, the scaling of CS levels in quiver diagram (a) does not give new toric data. Similarly, the scaling of CS levels in quiver diagram (b) gives the same toric data $\mathcal{G}_b(k)$ upto $GL(4, \mathbb{Z})$ transformation.
 - (ii) We can also deduce that the two quiver theories are distinct for $k \neq 1$ because the toric data of the two theories ($\mathcal{G}_a(k)$ and $\mathcal{G}_b(k)$) are not related by any $GL(4, \mathbb{Z})$ transformation.
 - (iii) When there is no orbifolding- i.e., $k = 1$, both the theories are same upto some $GL(4, \mathbb{Z})$ transformation. Hence these theories are actually just the phases of \mathbb{C}^4 theory, known as Phase-I and Phase-II of \mathbb{C}^4 respectively.
3. Theory with 2 bifundamentals X_{12}, X_{21} and adjoints ϕ_1, ϕ_2 at two different nodes with trivial $W = 0$. The quiver of this theory is shown in figure 3. This theory does

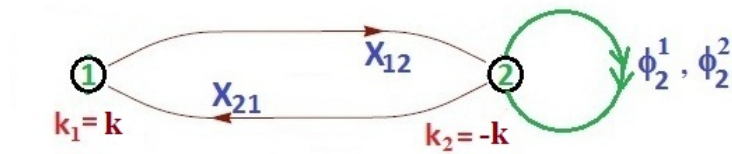


Figure 2. Quiver diagram (b)

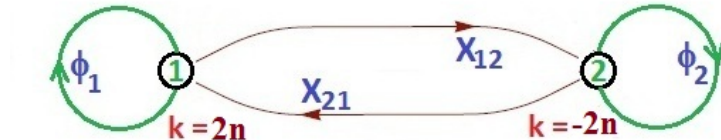


Figure 3. Quiver Diagram for Fano \mathbb{P}^3

not admit tiling and was first obtained using the inverse algorithm in [22], which was identified as the Fano \mathbb{P}^3 theory.

We cannot determine the CS levels from the inverse algorithm [22] for all the three 2-node quiver theories. From the tiling description, we can obtain the CS levels for the quiver diagrams (a) and (b). Comparing the inverse algorithm of the three theories, we had inferred that the theory corresponding to Fano \mathbb{P}^3 could have CS-levels $(4, -4)$. We will obtain the CS levels as $(2n, -2n)$ corresponding to \mathbb{P}^3 from the method of partial resolution where n is a non-zero integer.

The toric data for this theory is given as:

$$\mathcal{G}_{\mathbb{P}^3} = \begin{pmatrix} \frac{p_1}{1} & \frac{p_2}{1} & \frac{p_3}{1} & \frac{p_4}{1} & \frac{p_5}{1} \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}. \quad (2.3)$$

The charge matrix $Q_{\mathbb{P}^3}$ of this theory given by $Q_{\mathbb{P}^3} \cdot \mathcal{G}_{\mathbb{P}^3}^T = 0$ consists only of Q_F and is given by [22]:

$$Q_{\mathbb{P}^3} = Q_F = (1, 1, 1, 1, -4). \quad (2.4)$$

We would like to see these Calabi-Yau 4-folds, particularly Fano \mathbb{P}^3 , as embeddings inside Calabi-Yau 4-fold toric data corresponding to some 3-node quivers. In the following section, we will extensively discuss the 3-node quiver corresponding to Fano \mathcal{B}_4 which admits dimer tiling. Higgsing of the theory corresponding to Fano \mathcal{B}_4 [19], from the tiling approach,

has shown that the daughter theories are only quiver diagram (b). From the method of algebraic higgsing and partial resolutions, we get the toric data to be \mathbb{C}^4 and orbifolds of \mathbb{C}^4 . This will set the necessary tools and notations for studying the embeddings inside other three Fano \mathcal{B}_i 's in the later sections.

3 Fano \mathcal{B}_4

The quiver corresponding to the complex cone over Fano \mathcal{B}_4 is a theory with 3-nodes and 9 bifundamental fields $X_{12}^i, X_{23}^i, X_{31}^i$, where $i = 1, 2, 3$. This theory admits tiling and is known in the literature as Fano \mathbb{P}^3 or $M^{1,1,1}$ theory with CS-levels $\vec{k} = (1, -2, 1)$ [20]. Our aim is to show that this theory with scaled CS levels $n\vec{k}$, where n is a non-zero integer, corresponds to the Fano \mathcal{B}_4 toric data upto $GL(4, \mathbb{Z})$ for any n . This scaling freedom plays a crucial role in determining the CS levels of the daughter quiver obtained by higgsing. So, we shall prove the scaling freedom by using both forward and inverse algorithms in the following subsections. The quiver diagram for this theory is shown in figure 4. The superpotential of the theory is given by:

$$W = \text{Tr} \left[\epsilon_{ijk} X_{12}^i X_{23}^j X_{31}^k \right]. \quad (3.1)$$

3.1 Forward algorithm

The tiling approach of this theory has been studied in [20]. From the Kastelyne matrix, we can obtain toric data with scaled CS levels $n\vec{k}$. Though this toric data is dependent on n , we have checked that it is related to that of Fano \mathcal{B}_4 upto $GL(4, \mathbb{Z})$ transformation. For completeness, let us do the forward algorithm of the three-node quiver theory using scaled CS levels $(n, -2n, n)$. We will see that the toric data thus obtained is independent of the choice of n . From the quiver diagram in figure 4, the incidence matrix d representing the charge of the matter fields can be written as:

$$d = \left(\begin{array}{c|ccccccccc} & X_{12}^1 & X_{12}^2 & X_{12}^3 & X_{23}^1 & X_{23}^2 & X_{23}^3 & X_{31}^1 & X_{31}^2 & X_{31}^3 \\ \hline G=1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ G=2 & -1 & -1 & -1 & 1 & 1 & 1 & 0 & 0 & 0 \\ G=3 & 0 & 0 & 0 & -1 & -1 & -1 & 1 & 1 & 1 \end{array} \right), \quad (3.2)$$

where the rows indicate the gauge groups or the nodes in the quiver, and columns indicate the matter fields.

The projected charge matrix (Δ) will consist of a single row whose elements will be given by

$$\Delta_i = k_2 d_{1i} - k_1 d_{2i} = -n(2d_{1i} + d_{2i}). \quad (3.3)$$

Hence, Δ matrix will be given by:

$$\Delta = \left(\begin{array}{ccccccccc} X_{12}^1 & X_{12}^2 & X_{12}^3 & X_{23}^1 & X_{23}^2 & X_{23}^3 & X_{31}^1 & X_{31}^2 & X_{31}^3 \\ -n & -n & -n & -n & -n & -n & 2n & 2n & 2n \end{array} \right). \quad (3.4)$$

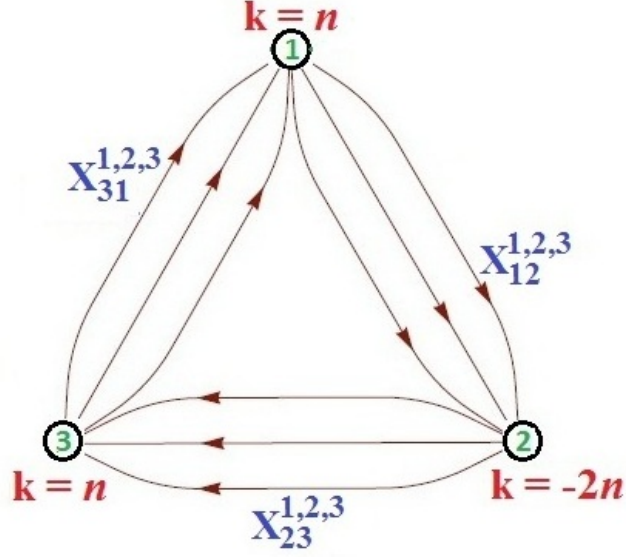


Figure 4. Quiver Diagram for Fano \mathcal{B}_4

From the superpotential W (3.1), one can find the F -term constraints given by the set of equations $\{\partial W/\partial X_i = 0\}$, which means that the matter fields X_i 's can be written in terms of 5 independent v -fields, and the relation between them can be encoded in a matrix K . The matrix K and its dual matrix T are given as:

$$K = \left(\begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline X_{12}^{(1)} & 0 & 0 & 1 & 0 & 0 \\ X_{12}^{(2)} & 0 & 0 & 1 & 0 & 1 \\ X_{12}^{(3)} & 0 & 0 & 1 & 1 & 0 \\ X_{23}^{(1)} & 0 & 1 & 0 & 0 & 0 \\ X_{23}^{(2)} & 0 & 1 & 0 & 0 & 1 \\ X_{23}^{(3)} & 0 & 1 & 0 & 1 & 0 \\ X_{31}^{(1)} & 1 & 0 & 0 & 0 & 0 \\ X_{31}^{(2)} & 1 & 0 & 0 & 0 & 1 \\ X_{31}^{(3)} & 1 & 0 & 0 & 1 & 0 \end{array} \right) ; T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} . \quad (3.5)$$

From K and T , we can write the matrix $P = K.T$. The entries of the matrix P are all

non-negative and it gives the relation of the matter fields X_i 's with the GLSM fields p_i 's:

$$P = \left(\begin{array}{c|cccccc} & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \\ \hline X_{12}^1 & 1 & 0 & 0 & 1 & 0 & 0 \\ X_{12}^2 & 0 & 1 & 0 & 1 & 0 & 0 \\ X_{12}^3 & 0 & 0 & 1 & 1 & 0 & 0 \\ X_{23}^1 & 1 & 0 & 0 & 0 & 1 & 0 \\ X_{23}^2 & 0 & 1 & 0 & 0 & 1 & 0 \\ X_{23}^3 & 0 & 0 & 1 & 0 & 1 & 0 \\ X_{31}^1 & 1 & 0 & 0 & 0 & 0 & 1 \\ X_{31}^2 & 0 & 1 & 0 & 0 & 0 & 1 \\ X_{31}^3 & 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right). \quad (3.6)$$

The Q_F charge matrix is given by the nullspace of P :

$$Q_F = \begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 \end{pmatrix}. \quad (3.7)$$

The steps done so far in obtaining P -matrix and the charge Q_F are independent of the CS levels. The information about CS levels is contained only in the charge Q_D matrix which is a single row obeying the symmetry of the Calabi-Yau whose general form can be written as [22] $Q_D = \begin{pmatrix} a_1 & a_1 & a_1 & a_2 & a_2 & a_3 \end{pmatrix}$. The relation between P , Δ and Q_D is given by the equation,

$$\Delta_{bi} = P_{ic}(Q_D)_{bc}. \quad (3.8)$$

Using this equation and eqn.(3.4), a possible form for Q_D matrix is

$$Q_D = \begin{bmatrix} 0 & 0 & 0 & -n & -n & 2n \end{bmatrix}. \quad (3.9)$$

The total charge matrix can be obtained by concatenating (3.7) and (3.9) in a single matrix Q :

$$Q = \begin{pmatrix} Q_F \\ Q_D \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -n & -n & 2n \end{pmatrix}. \quad (3.10)$$

The toric data $\mathcal{G}(n, -2n, n)$ will be given by nullspace of Q :

$$\mathcal{G}(n, -2n, n) = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \\ \hline 3 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.11)$$

where we have used the notation $\mathcal{G}(n, -2n, n)$ to specify that this is the toric data for the CS levles $(n, -2n, n)$. Clearly, the toric data is independent of n . From the Kastelyne

matrix in the tiling method, the toric data for the level $(1, -2, 1)$ is[20]:

$$\mathcal{G}(1, -2, 1) = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}. \quad (3.12)$$

These two toric datas are related by $GL(4, \mathbb{Z})$ transformation:

$$\mathcal{G}(n, -2n, n) = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \cdot \mathcal{G}(1, -2, 1). \quad (3.13)$$

Thus we see that we have a scaling freedom in the choice of the CS levels for the 3-node quiver giving the same \mathcal{B}_4 toric data. This freedom can also be seen in the inverse algorithm. Note that the inverse algorithm for Fano \mathcal{B}_4 has been done in[22]. We shall quickly repeat the steps to stress the scaling freedom of the CS levels.

3.2 Inverse algorithm

Here we start with the toric data of \mathcal{B}_4 which is given by eqn.(3.12). From the ansatz given in[22], Q_F in eqn.(3.7) is a possible choice. The matrices $T = \text{Nullspace}(Q_F)$ and $K = \text{dual}(T)$ for this choice of Q_F will reproduce the same T and K in eqn.(3.5). The Q_F (3.7) gives the K -matrix which determines superpotential W and the number of matter fields. This data allows only one possible quiver upto relabeling of the nodes as shown in figure 4. A possible choice of Q_D consistent with the ansatz[22] can be taken as eqn.(3.9). Using the matrix $P = K.T$ (3.6) and the eqns.(3.3,3.8), we can obtain the Δ and the CS levels. The CS levels come out to be $(k_1, k_2, k_3) = (n, -2n, n)$. The steps of the inverse algorithm confirm that the \mathcal{B}_4 toric data is respected by quiver diagram with CS levels scaled by a factor n .

This exercise of the forward and the inverse algorithm for Fano \mathcal{B}_4 shows that the charge Q_F , P and the quiver charge matrix d are unaffected by scaling of CS levels: $\vec{k} \rightarrow n\vec{k}$. It is only the Δ and Q_D matrix which gets multiplied by n satisfying eqn.(3.8). For the 2-node quivers discussed in the previous section 2, $Q_D = 0$ and hence the scaling freedom of CS levels cannot be seen from the inverse algorithm. Therefore we have to study the higgsing or partial resolution to get the CS levels and the scaling freedom for the two-node quiver corresponding to Fano \mathbb{P}^3 which has non-trivial Q_F matrix.

From the tiling approach, higgsing of the quiver diagram in figure 4 for $n = 1$ has been presented in detail in Ref.[19]. With the scaling freedom of CS levels of parent quiver, we can obtain daughter quivers in figure 2 corresponding to \mathbb{C}^4 and \mathbb{Z}_n orbifolds of \mathbb{C}^4 . We

will now study the algebraic approach of higgsing which will be applicable for other quivers that do not admit dimer tiling presentation.

3.3 Algebraic higgsing

We attempt the higgsing of Fano \mathcal{B}_4 theory to obtain one of the 2 nodes theories discussed in section 2. In algebraic higgsing, we choose some matter field, say X_i and give a non zero VEV to it. Giving a VEV makes the matter field massive and hence removed from the quiver. However, in the process of giving VEV, all those GLSM p_j fields which contain the matter field X_i also become massive and hence must be removed. To do this, we delete the i -th row from the P matrix which corresponds to the matter field X_i and also, we remove all those columns (p -fields) which have non-zero entry corresponding to i -th row (and hence have become massive).

As an example, let us take the X_{12}^1 field of the \mathcal{B}_4 theory and give a VEV to it. Thus, from the P matrix (3.6), we must remove the first row and also the columns 1 and 4 which correspond to the GLSM fields p_1, p_4 which contain the X_{12}^1 field. After giving VEV to the X_{12}^1 field, the nodes 1 and 2 in the figure 4 are collapsed giving a 2 node daughter theory with CS levels $(n, -n)$. Removal of the corresponding row and columns in eqn.(3.6) will give the following reduced P -matrix:

$$P_r = \left(\begin{array}{c|cccc} & p_2 & p_3 & p_5 & p_6 \\ \hline X_{12}^2 & 1 & 0 & 0 & 0 \\ X_{12}^3 & 0 & 1 & 0 & 0 \\ X_{23}^1 & 0 & 0 & 1 & 0 \\ X_{23}^2 & 1 & 0 & 1 & 0 \\ X_{23}^3 & 0 & 1 & 1 & 0 \\ X_{31}^1 & 0 & 0 & 0 & 1 \\ X_{31}^2 & 1 & 0 & 0 & 1 \\ X_{31}^3 & 0 & 1 & 0 & 1 \end{array} \right). \quad (3.14)$$

The nullspace of this matrix gives the reduced charge matrix $Q_{F_r} = 0$. Thus, we see that the Q_{F_r} of the daughter theory is trivial. The quivers (a) and (b) listed in section 2 have $Q_F = 0$. So, we claim that the algebraic higgsing gives daughter quivers (a) as well as quiver (b) with CS levels $(n, -n)$.

By giving VEV to any other matter fields, we have checked that we get the same trivial $Q_{F_r} = 0$. This suggests that the algebraic higgsing of the parent quiver corresponding to Fano \mathcal{B}_4 will give quivers corresponding to \mathbb{C}^4 and orbifolds of \mathbb{C}^4 . It must be mentioned at this point that the algebraic higgsing of Fano $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ also gives a daughter theory with trivial Q_F . So, the algebraic higgsing tells that only \mathbb{C}^4 or the orbifolds of \mathbb{C}^4 are embedded inside Fano $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4$. It may be possible that we may get non-trivial Q_F from the method of partial resolution. So, we shall study the method of partial resolution for Fano \mathcal{B}_4 toric data.

3.4 Partial Resolution

In partial resolution, we try to remove the points from the toric diagram itself. The resulting toric diagram corresponds to some daughter theory which is embedded in the parent theory. From the toric diagram of \mathcal{B}_4 , we will remove some points which amounts to removing the corresponding columns (or the corresponding p -fields) from the toric data \mathcal{G} . Next, we will check whether this reduced toric data (denoted as \mathcal{G}_r), which is obtained by removing the columns from original \mathcal{G} , is related to any of the toric data of the 2-node theories.

Take the toric data \mathcal{G} of Fano \mathcal{B}_4 in eqn.(3.12). We see that if we remove the points p_1 and p_4 , we get the reduced toric data

$$\mathcal{G}_r = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \quad (3.15)$$

Both $\mathcal{G}_a(k)$ (2.1) and $\mathcal{G}_b(k)$ (2.2) for any general k are $GL(4, \mathbb{Z})$ related to \mathcal{G}_r :

$$\mathcal{G}_a(k) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -k \end{pmatrix} \cdot \mathcal{G}_r, \quad (3.16)$$

$$\mathcal{G}_b(k) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -k \end{pmatrix} \cdot \mathcal{G}_r. \quad (3.17)$$

Alternatively, we can take the charge matrix Q of the \mathcal{B}_4 theory given in eqn.(3.10) and take a linear combination and set the charge matrix elements corresponding to column p_1 and p_4 to zero as described below: Suppose that a row (r) of Q of the daughter theory is given as some linear combination of rows (R_i) of Q (3.10) of parent theory, i.e.

$$r = \sum_{i=1}^2 (a_i R_i) = (a_1, a_1, a_1, -a_1 - na_2, -a_1 - na_2, -a_1 + 2na_2). \quad (3.18)$$

Since we are removing p_1 and p_4 points, the corresponding columns in r are set to 0 and also removed. This will give the values of a_1 and a_2 , and we find that $a_1 = a_2 = 0$. Hence, the Q of the daughter theory is trivial.

We have obtained \mathcal{G}_r by removing all other possible set of points in the toric data of \mathcal{B}_4 - namely., $\{p_1, p_5\}$, $\{p_2, p_4\}$, $\{p_2, p_5\}$, $\{p_3, p_4\}$ and $\{p_4, p_5\}$ and checked that they are again related by $GL(4, \mathbb{Z})$ to $\mathcal{G}_a(k)$ as well as to $\mathcal{G}_b(k)$. For all these cases, the linear

combination of the charge Q matrix (3.10) with the appropriate columns removed gives Q of the daughter theory to be trivial ($Q = 0$).

Thus we see that only \mathbb{C}^4 and all the orbifolds of \mathbb{C}^4 are embedded in the \mathcal{B}_4 theory. Hence, partial resolution allows both the quivers (a) and (b) as possible quivers for the daughter theory confirming the results of algebraic higgsing.

The quiver gauge theories corresponding to Fano \mathcal{B}_3 and \mathcal{B}_2 have non-trivial superpotential W [22] and hence can be described by both forward and inverse algorithm. Similar to Fano \mathcal{B}_4 , we can show the scaling of CS levels implies scaling of Q_D charge matrix which gives the same toric data. We will briefly present in the next section, the quiver and necessary data for Fano \mathcal{B}_3 and study algebraic higgsing and partial resolution.

4 Fano \mathcal{B}_3

It is a theory with 3-nodes, 2 adjoint fields X_1, X_2 on node-1 and 6 bifundamental fields $X_3, X_4, X_5, X_6, X_7, X_8$. The quiver diagram for this theory is shown in figure 5. This theory does not admit tiling and was first studied in [22], where it was identified as the quiver gauge theory for Fano \mathcal{B}_3 . In [22], it was mentioned that the CS-levels of this theory are (6,-6,0). However, following the arguments in the previous section, it is straightforward to see that the theory allows the CS levels $(6n, -6n, 0)$, where n is non-zero integer. The superpotential of the theory is given by:

$$W = \text{Tr} [(X_1 X_4 - X_2 X_3) (X_5 X_8 - X_6 X_7)] . \quad (4.1)$$

From W given in 4.1, we construct K , which gives T and hence P :

$$K = \left(\begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline X_1 & 1 & 0 & 0 & 0 & 0 \\ X_2 & 1 & 0 & 0 & 0 & 3 \\ X_3 & 1 & 0 & 0 & 1 & 0 \\ X_4 & 1 & 0 & 0 & 1 & 3 \\ X_5 & 1 & 0 & 3 & 0 & 0 \\ X_6 & 1 & 0 & 3 & 4 & 0 \\ X_7 & 1 & 3 & 0 & 0 & 0 \\ X_8 & 1 & 3 & 0 & 4 & 0 \end{array} \right) ; T = \begin{pmatrix} 3 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 0 & 2 & 0 \\ -1 & 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} ; P = \left(\begin{array}{c|ccccccc} & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 \\ \hline X_1 & 3 & 0 & 1 & 0 & 0 & 0 & 1 \\ X_2 & 0 & 3 & 1 & 0 & 0 & 0 & 1 \\ X_3 & 3 & 0 & 0 & 1 & 0 & 0 & 1 \\ X_4 & 0 & 3 & 0 & 1 & 0 & 0 & 1 \\ X_5 & 0 & 0 & 4 & 0 & 6 & 0 & 1 \\ X_6 & 0 & 0 & 0 & 4 & 6 & 0 & 1 \\ X_7 & 0 & 0 & 4 & 0 & 0 & 6 & 1 \\ X_8 & 0 & 0 & 0 & 4 & 0 & 6 & 1 \end{array} \right) . \quad (4.2)$$

The incidence matrix d for the quiver in figure 5 is given by,

$$d = \left(\begin{array}{c|cccc} & X_i(i=1,2) & X_i(i=3,4) & X_i(i=5,6) & X_i(i=7,8) \\ \hline G=1 & 0 & 1 & 0 & -1 \\ G=2 & 0 & -1 & 1 & 0 \\ G=3 & 0 & 0 & -1 & 1 \end{array} \right) . \quad (4.3)$$

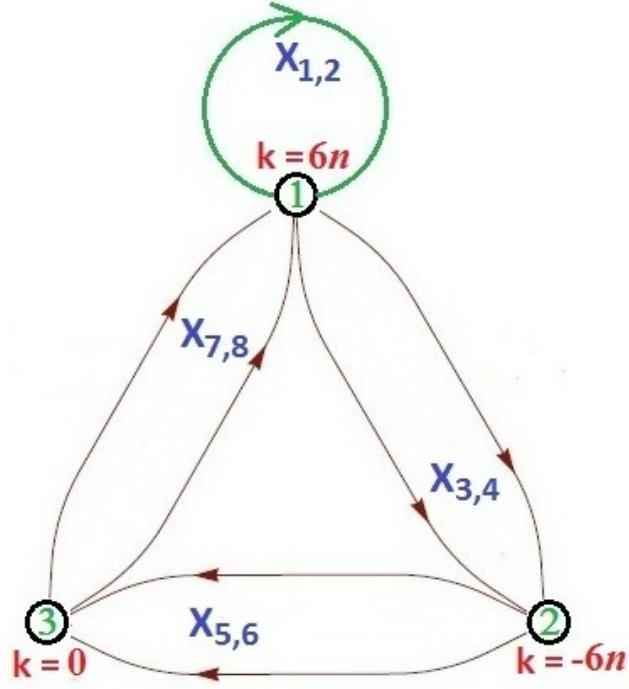


Figure 5. Quiver Diagram for Fano \mathcal{B}_3

The projected charge matrix (Δ) is given by:

$$\Delta = \left(\begin{array}{c|c|c|c} X_i(i=1,2) & X_i(i=3,4) & X_i(i=5,6) & X_i(i=7,8) \\ \hline 0 & 0 & -6n & 6n \end{array} \right). \quad (4.4)$$

Using the eqn.(3.8), the general form of Q_D charge matrix obeying the symmetry of Calabi-Yau over Fano \mathcal{B}_3 is given by:

$$Q_D = \begin{pmatrix} 0 & 0 & 2n & 2n & -2n & 0 & -2n \end{pmatrix}. \quad (4.5)$$

The Q_F charge matrix is given by the nullspace of P :

$$Q_F = \begin{pmatrix} 1 & 1 & 3 & 3 & -1 & -1 & -6 \\ 1 & 1 & 1 & 1 & 0 & 0 & -4 \end{pmatrix}. \quad (4.6)$$

Total charge matrix Q will be given by:

$$Q = \begin{pmatrix} Q_F \\ Q_D \end{pmatrix} = \begin{pmatrix} 1 & 1 & 3 & 3 & -1 & -1 & -6 \\ 1 & 1 & 1 & 1 & 0 & 0 & -4 \\ 0 & 0 & 2n & 2n & -2n & 0 & -2n \end{pmatrix}. \quad (4.7)$$

Thus the toric data for \mathcal{B}_3 theory for CS levels $(6n, -6n, 0)$ is given by:

$$\mathcal{G}(6n, -6n, 0) = \begin{pmatrix} \frac{p_1}{3} & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.8)$$

Also, the toric data for CS levels $(6, -6, 0)$ is given as [22]:

$$\mathcal{G}(6, -6, 0) = \begin{pmatrix} \frac{p_1}{1} & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & -1 & 0 \end{pmatrix}. \quad (4.9)$$

We find that these two toric datas are related by $GL(4, \mathbb{Z})$ transformation:

$$\mathcal{G}(6n, -6n, 0) = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \cdot \mathcal{G}(6, -6, 0). \quad (4.10)$$

Similarly, we can start from the toric data (4.9) and take the charge matrix Q as given in eqn.(4.7) consistent with the ansatz [22]. Following the steps of inverse algorithm we can obtain the quiver as shown in figure 5 and using eqn.(3.8) we obtain the CS levels as $(6n, -6n, 0)$. We will now study algebraic higgsing and obtain two-node daughter quivers.

4.1 Algebraic higgsing

If we give a VEV to any of the X_i fields, thereby removing the corresponding row and columns from the P -matrix(4.2), we see that we get a reduced matrix (P_r) , whose null space (Q_{F_r}) is always trivial. Starting from the parent quiver as shown in figure 5, we see that the CS levels of the 2-node daughter quiver can be either $(0, 0)$ or $(6n, -6n)$. Thus, we can only say that the higgsing of the Fano \mathcal{B}_3 theory will give either \mathbb{C}^4 or the \mathbb{Z}_{6n} orbifolds of \mathbb{C}^4 as the daughter theory. Moreover, we see that the last column (p_7) of P matrix(4.2) which corresponds to the internal point in the toric diagram of Fano \mathcal{B}_3 will always be removed because it contains all the matter fields. So, this method cannot give a daughter theory corresponding to Fano \mathbb{P}^3 which has an internal point in the toric diagram. Hence we are forced to study the method of partial resolution to check whether toric Fano \mathbb{P}^3 is embedded inside \mathcal{B}_3 Fano.

4.2 Partial Resolution

Here, we do the partial resolution of Fano \mathcal{B}_3 theory and check whether we get non-trivial charge Q for the daughter theory.

4.2.1 Embedding of Fano \mathbb{P}^3 inside Fano \mathcal{B}_3

It is interesting to see that the method of partial resolution does embed the toric Fano \mathbb{P}^3 inside Fano \mathcal{B}_3 giving the correct Q_F (2.4). Hence we can claim that this method gives more information than algebraic higgsing for quiver which do not admit dimer tiling presentation. Suppose we remove the points $\{p_5, p_6\}$ from the \mathcal{G} (4.9) we get the reduced toric data

$$\mathcal{G}_r = \begin{pmatrix} \frac{p_1}{1} & \frac{p_2}{1} & \frac{p_3}{1} & \frac{p_4}{1} & \frac{p_7}{1} \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{pmatrix} \quad (4.11)$$

and the toric data $\mathcal{G}_{\mathbb{P}^3}$ (2.3) is related to \mathcal{G}_r by a $GL(4, \mathbb{Z})$ transformation:

$$\mathcal{G}_{\mathbb{P}^3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \mathcal{G}_r . \quad (4.12)$$

A row (r) of Q of the daughter theory will be given as linear combination of rows (R_i) of Q (4.7) i.e.,

$$r = \sum_{i=1}^3 (a_i R_i) = (a_1 + a_2, a_1 + a_2, 3a_1 + a_2 + 2na_3, 3a_1 + a_2 + 2na_3, -a_1 - 2na_3, -a_1, -6a_1 - 4a_2 - 2na_3) .$$

Setting the columns 5,6 in r to 0 gives $a_1 = a_3 = 0$. Removing these columns gives the reduced charge matrix as:

$$r = (a_2, a_2, a_2, a_2, -4a_2) = a_2(1, 1, 1, 1, -4) .$$

Thus, Q will have only one row generated by $(1, 1, 1, 1, -4)$. Hence, the charge matrix of daughter theory is same as charge matrix of the Fano \mathbb{P}^3 (2.4) upto an overall scaling a_2 . Thus, we see that Fano \mathbb{P}^3 theory is embedded in the Fano \mathcal{B}_3 theory. As the algebraic higgsing was unable to predict this embedding, we cannot precisely say which nodes in the 3-node quiver coalesced. However comparing the parent quiver (figure 5) and the daughter quiver (figure 3), we can indirectly say that the daughter theory is obtained by collapsing either the nodes-1,3 or the nodes-2,3 in figure 5. So the CS levels of the 2-node daughter quiver corresponding to the Fano \mathbb{P}^3 theory will be $(6n, -6n)$. We shall come to this point again when we discuss the partial resolution of Fano \mathcal{B}_2 theory.

4.2.2 Embedding of \mathbb{C}^4 and orbifolds of \mathbb{C}^4 inside Fano \mathcal{B}_3

If we remove points $\{p_1, p_3, p_5\}$ from the toric data(4.9) we will get a reduced toric data given by:

$$\mathcal{G}_r = \begin{pmatrix} \frac{p_2}{1} & \frac{p_4}{1} & \frac{p_6}{1} & \frac{p_7}{1} \\ 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & -1 & -1 & 0 \end{pmatrix} .$$

Both $\mathcal{G}_a(k)$ and $\mathcal{G}_b(k)$ are related to it:

$$\mathcal{G}_a(k) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -k & k & -k \end{pmatrix} \cdot \mathcal{G}_r, \quad (4.13)$$

$$\mathcal{G}_b(k) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -k & k & -k \end{pmatrix} \cdot \mathcal{G}_r. \quad (4.14)$$

Also, we can find the reduced charge matrix Q_r in a similar way as was done for other cases, and we find that it is trivial. Similarly, if we remove the other set of points $\{p_1, p_4, p_5\}$, $\{p_2, p_3, p_5\}$ and $\{p_2, p_4, p_5\}$, we will get trivial Q_r . Indirectly by coalescing any of the two nodes in the parent quiver, we would get both \mathbb{C}^4 and \mathbb{Z}_{6n} orbifolds of \mathbb{C}^4 as embedding inside \mathcal{B}_3 . In the following section, We will briefly present the quiver corresponding to Fano \mathcal{B}_2 and study the partial resolution.

5 Fano \mathcal{B}_2

The quiver Chern-Simons theory corresponding to Fano \mathcal{B}_2 [22] has 3 nodes, 12 matter fields with 2 possible quiver diagrams as shown in figure 6 and figure 7. The superpotential of the theory is given by:

$$W = \text{Tr}(X_1 X_4 X_8 X_{12} - X_1 X_4 X_9 X_{11} - X_2 X_5 X_7 X_{12} + X_2 X_5 X_9 X_{10} + X_3 X_6 X_7 X_{11} - X_3 X_6 X_8 X_{10}). \quad (5.1)$$

This theory does not admit dimer tiling presentation but can be studied using forward algorithm. So, following similar arguments in the previous section, we get the same \mathcal{B}_2 toric data (upto $GL(4, \mathbb{Z})$ transformation) if we scale the CS levels $(2, -2, 0) \rightarrow n(2, -2, 0)$ in both the 3-node quivers (figure 6&7). Here n is a non-zero integer. Using W given in eqn.(5.1), one can construct K which gives T whence P :

$$P = \left(\begin{array}{c|cccccccc} & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 \\ \hline X_1 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 1 \\ X_2 & 0 & 2 & 0 & 2 & 2 & 0 & 0 & 1 \\ X_3 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 1 \\ X_4 & 2 & 0 & 0 & 0 & 2 & 2 & 0 & 1 \\ X_5 & 0 & 2 & 0 & 0 & 2 & 2 & 0 & 1 \\ X_6 & 0 & 0 & 2 & 0 & 2 & 2 & 0 & 1 \\ X_7 & 4 & 0 & 0 & 2 & 0 & 0 & 2 & 1 \\ X_8 & 0 & 4 & 0 & 2 & 0 & 0 & 2 & 1 \\ X_9 & 0 & 0 & 4 & 2 & 0 & 0 & 2 & 1 \\ X_{10} & 4 & 0 & 0 & 0 & 0 & 2 & 2 & 1 \\ X_{11} & 0 & 4 & 0 & 0 & 0 & 2 & 2 & 1 \\ X_{12} & 0 & 0 & 4 & 0 & 0 & 2 & 2 & 1 \end{array} \right). \quad (5.2)$$

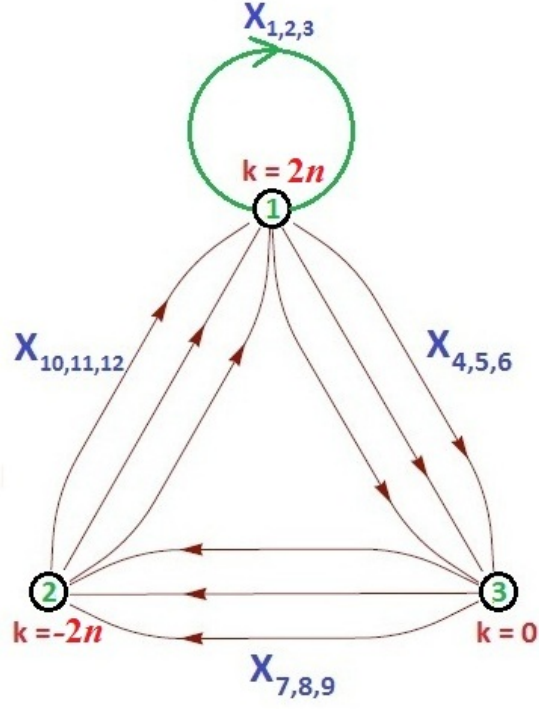


Figure 6. Cyclic quiver for Fano \mathcal{B}_2

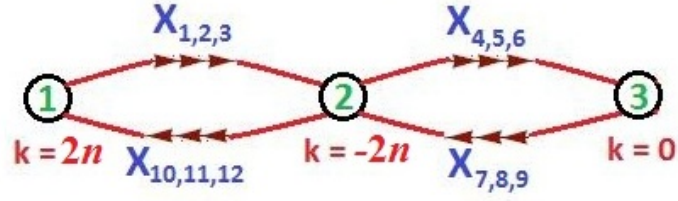


Figure 7. Linear quiver for Fano \mathcal{B}_2

A possible choice of the total charge matrix Q is given by:

$$Q = \begin{pmatrix} \frac{Q_F}{Q_D} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & n & n & 2n & 0 & -4n \end{pmatrix}. \quad (5.3)$$

The toric data for \mathcal{B}_2 theory for CS levels $(2n, -2n, 0)$ will turn out to be:

$$\mathcal{G}(2n, -2n, 0) = \begin{pmatrix} \frac{p_1}{2} & \frac{p_2}{0} & \frac{p_3}{0} & \frac{p_4}{2} & \frac{p_5}{2} & \frac{p_6}{0} & \frac{p_7}{0} & \frac{p_8}{1} \\ 1 & 0 & 0 & -1 & -1 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (5.4)$$

Also, the toric data for CS levels $(2, -2, 0)$ is given as[22]:

$$\mathcal{G}(2, -2, 0) = \begin{pmatrix} \frac{p_1}{1} & \frac{p_2}{1} & \frac{p_3}{1} & \frac{p_4}{1} & \frac{p_5}{1} & \frac{p_6}{1} & \frac{p_7}{1} & \frac{p_8}{1} \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 1 & 0 \end{pmatrix}. \quad (5.5)$$

We find that these two are related by $GL(4, \mathbb{Z})$ transformation:

$$\mathcal{G}(2n, -2n, 0) = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \cdot \mathcal{G}(2, -2, 0). \quad (5.6)$$

Starting from the toric data in the the inverse algorithm for \mathcal{B}_2 theory[22], it is a straightforward exercise to show the scaling freedom in the CS levels. Taking the P matrix (5.2), giving VEV to any of the matter fields gives $Q_{F_r} = 0$. Also from the quiver diagrams given in figure 6 and figure 7, we know that any 2 node daughter theory will have the CS levels either $(0, 0)$ or $(2n, -2n)$. Thus, the daughter theory will be either \mathbb{C}^4 or the \mathbb{Z}_{2n} orbifolds of \mathbb{C}^4 . From eqn.(5.2) we see that giving a VEV to any of the matter fields will always remove the last column (p_8) of the P which corresponds to an internal point in the toric diagram of \mathcal{B}_2 . So if we do the algebraic higgsing of \mathcal{B}_2 , we are never going to get the embedding as Fano \mathbb{P}^3 which has an internal point in its toric diagram. We will now work out the partial resolution to see if we can get more information.

5.1 Partial Resolution

In this case, we found that the if we remove the set of points $\{p_1, p_4, p_5, p_7\}$, $\{p_1, p_4, p_6, p_7\}$, $\{p_2, p_4, p_5, p_7\}$, $\{p_2, p_4, p_6, p_7\}$, $\{p_3, p_4, p_5, p_7\}$, $\{p_3, p_4, p_6, p_7\}$ or $\{p_4, p_5, p_6, p_7\}$, we will get a reduced toric data whose nullspace, i.e. the reduced charge matrix Q_r is trivial. This implies that the toric data of the daughter theory is either \mathbb{C}^4 or some orbifolding of it. Looking at the quiver, we can say that either \mathbb{C}^4 or \mathbb{Z}_{2n} orbifolds of \mathbb{C}^4 are embedded inside Fano \mathcal{B}_2 , confirming the result of algebraic higgsing.

5.1.1 Embedding of Fano \mathbb{P}^3 inside Fano \mathcal{B}_2

If we remove the points $\{p_5, p_6, p_7\}$ from the toric data \mathcal{G} given in eqn.(5.5), we will get the reduced toric data:

$$\mathcal{G}_r = \begin{pmatrix} \frac{p_1}{1} & \frac{p_2}{1} & \frac{p_3}{1} & \frac{p_4}{1} & \frac{p_8}{1} \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix},$$

which is exactly same as the toric data of \mathbb{P}^3 . Thus we see that \mathbb{P}^3 is embedded inside \mathcal{B}_2 .

Taking a row (r) of total charge Q of the daughter theory as a linear combination of rows (R_i) of total charge matrix of \mathcal{B}_2 given in eqn.(5.3):

$$r = (a_1, a_1, a_1, -a_1 + a_2 + na_4, -a_1 - a_2 + a_3 + na_4, -a_1 + a_2 + 2na_4, -2a_1 - a_2 + a_3, 2a_1 - 2a_3 - 4na_4) .$$

Setting the columns 5,6,7 in r to 0, we get $a_1 = a_2/3 = a_3/5 = -a_4$. The reduced charge matrix after removal of the columns 5,6,7 gives:

$$r = (a_1, a_1, a_1, a_1, -4a_1) = a_1(1, 1, 1, 1, -4) .$$

Thus, Q will have only 1 row generated by $(1, 1, 1, 1, -4)$. Hence, the charge matrix of daughter theory is $Q = (Q_F) = (1, 1, 1, 1, -4)$ which is the charge matrix of \mathbb{P}^3 .

Now, let us compare this result with that discussed in the partial resolution of Fano \mathcal{B}_3 to \mathbb{P}^3 in section-4. In the \mathcal{B}_3 case, we saw that the CS-levels for Fano \mathbb{P}^3 to be $(6n, -6n)$. However, here we see that they must be $(2n, -2n)$. Incorporating the scaling freedom of CS levels, the partial resolution method fixes the CS-levels for Fano \mathbb{P}^3 theory to $(2n, -2n)$. Moreover, in [22], the CS-levels proposed as $(4, -4)$ is consistent with $(2n, -2n)$. We will now study partial resolution of Fano \mathcal{B}_a in the following section.

6 Fano \mathcal{B}_1

This theory was studied in [22] where the inverse algorithm was used to find a quiver gauge theory shown in figure 8. The superpotential of this theory was obtained as:

$$W = \text{Tr} [X_2 X_5 X_8 (X_1 X_4 X_9 X_3 X_6 X_7 - X_1 X_6 X_7 X_3 X_4 X_9)] . \quad (6.1)$$

Note that the abelian $W = 0$ and the quiver does not admit tiling. So, there is no way to obtain the toric data of this theory using forward algorithm or dimer tiling approach. But we can infer the scaling freedom of CS levels from the inverse algorithm. The toric data for \mathcal{B}_1 theory with multiplicity is given as [22]:

$$\mathcal{G} = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 & 0 \end{pmatrix} . \quad (6.2)$$

From the ansatz given in [22], we can take a choice of Q_F and Q_D obeying the symmetry of Calabi-Yau over Fano \mathcal{B}_1 and the total charge matrix will be given by:

$$Q = \begin{pmatrix} Q_F \\ Q_D \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & -2 & -2 & -2 & 3 \\ 0 & 0 & 0 & 2 & 1 & 1 & -4 \\ 0 & 0 & 0 & n & 0 & n & -2n \end{pmatrix} . \quad (6.3)$$

Using these, we can find the quiver theory which is given by the quiver in figure 8 with a scaling freedom in CS levels given by $(2n, 0, -2n)$, where n is a non-zero integer.

The algebraic higgsing in this case gives \mathbb{C}^4 or the \mathbb{Z}_{2n} orbifolds of \mathbb{C}^4 and we also observe that the internal point in the toric Fano \mathcal{B}_1 gets removed. We get the same information from the partial resolution by removing the points $\{p_1, p_4, p_5\}$, $\{p_2, p_4, p_5\}$ or $\{p_3, p_4, p_5\}$. Hence we can conclude that \mathbb{P}^3 is not embedded into \mathcal{B}_1 .

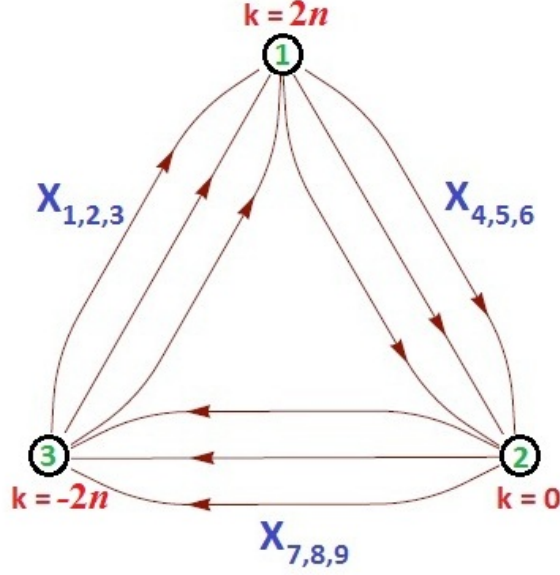


Figure 8. Cyclic quiver for Fano \mathcal{B}_1

7 Conclusions

Our main motivation was to determine the 2-node daughter quivers by the method of higgsing the 3-node parent quivers. Particularly, we wanted to higgsise matter fields of the 3-node quivers corresponding to Fano $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4$ 3-folds and obtain the daughter quivers. This procedure will determine the CS levels of the daughter quiver theories. Unfortunately, the algebraic method of higgsing does not give any non-trivial charge matrix Q .

We can perform forward algorithm for any quiver diagram with a different set of CS levels and obtain in general a new toric data. However if we scale the CS levels of the quiver by a non-zero integer n ($\vec{k} \rightarrow n\vec{k}$), we end up getting the same toric data upto $GL(4, \mathbb{Z})$ transformation. We have illustrated the scaling freedom of the CS levels for the 3-node quiver corresponding to Fano \mathcal{B}_4 toric data. This scaling freedom of the parent quiver dictates the CS levels of the daughter quivers.

For the quiver corresponding to Fano \mathcal{B}_4 , which has dimer tiling presentation, higgsing could be studied by tiling approach as well as by algebraic method. By the algebraic method of higgsing, we get the reduced charge matrix $Q_r = 0$ which suggests that the two-node daughter theories can be \mathbb{C}^4 and orbifolds of \mathbb{C}^4 . We further verified this result by the method of partial resolution of the toric data corresponding to Fano \mathcal{B}_4 .

The 3-node quivers for other Fano \mathcal{B} which were determined from the inverse algorithm do not admit dimer tiling presentation. So, we cannot study higgsing of these theories from tiling approach. The algebraic higgsing of any matter field removes the information of the internal point in the toric data and always gives trivial reduced charge matrix $Q_r = 0$. Unlike \mathbb{C}^4 and its Z_n orbifolds, the toric data of Fano \mathbb{P}^3 has an internal point. So, we studied the method of partial

resolution for toric Fano $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ 3-folds.

We found that the Fano \mathbb{P}^3 can be embedded inside Fano \mathcal{B}_3 . From the parent quiver CS levels including the scaling freedom of CS levels, we can infer the CS levels of the daughter quiver corresponding to Fano \mathbb{P}^3 to be $(6n, -6n)$. From the partial resolution of toric Fano \mathcal{B}_2 , we again obtained toric Fano \mathbb{P}^3 suggesting the CS levels of the 2-node quiver must be $(2n, -2n)$.

Taking into account the two possible CS levels of the two-node quiver, corresponding to Fano \mathbb{P}^3 , obtained from parent quivers corresponding to Fano \mathcal{B}_3 and \mathcal{B}_2 , we conclude $n(2, -2)$ as the scaled CS levels.

Algebraic higgsing and unhiggsing of quiver theories corresponding to some Fano 3-folds have been studied recently[24]. Higgsing certain matter fields in the 4-node quivers corresponding to toric Fano \mathcal{C}_4 gives the 3-node quiver corresponding to Fano \mathcal{B}_4 . Also higgsing of the quiver corresponding to Fano \mathcal{D}_2 gives daughter quiver corresponding to Fano \mathcal{B}_4 . These results can also be reproduced using the method of partial resolution.

It is not obvious whether we can obtain other Fano \mathcal{B} toric data as embeddings inside Fano \mathcal{C}_4 and Fano \mathcal{D}_2 3-folds. One of the issues is about our choice of multiplicity of certain points in the toric data for Fano $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$.

We had chosen a charge matrix Q_F, Q_D respecting the ansatz[22] which determined the multiplicity of certain points in the toric data of Fano \mathcal{B} 's. The toric data with the specific multiplicity of certain points was important to obtain sensible quivers. In principle, we would like to do the method of partial resolution for toric data corresponding to some 4-node quivers and reproduce the toric data of Fano $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ with the correct multiplicity of some of the points. We hope to study in future the embeddings of the toric Fano \mathcal{B} 's inside toric four-folds corresponding to 4-node quiver Chern-Simons theories.

References

- [1] J.Bagger and N.Lambert, “Modeling multiple M2’s,” Phys. Rev. **D75**, 045020 (2007) [arXiv:hep-th/0611108].
- [2] J.Bagger and N.Lambert, “Gauge symmetry and Supersymmetry of Multiple M2-Branes,” Phys. Rev. **D77**, 065008 (2008) [arXiv:0711.0955[hep-th]].
- [3] J.Bagger and N.Lambert, “Comments on Multiple M2-branes,” JHEP **0802**, 105 (2008) [arXiv:0712.3738[hep-th]].
- [4] A.Gustavsson, “Algebraic structures on parallel M2-branes,” Nucl.Phys. **B811**, 66 (2009) [arXiv:0709.1260[hep-th]].
- [5] A.Gustavsson, “Selfdual strings and loop space Nahm equations,” JHEP **0804**, 083 (2008) [arXiv:0802.3456[hep-th]].
- [6] M.Van Raamsdonk, “Comments on the Bagger-Lambert theory and multiple M2-branes,” JHEP **0805**, 105 (2008) [arXiv:0803.3803].
- [7] O.Aharony, O. Bergman, D.L. Jafferis and J. Maldacena, “N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals,” JHEP **0810**, 091 (2008) [arXiv:0806.1218[hep-th]].
- [8] I.R.Klebanov and G.Torri, “M2-branes and AdS/CFT,” Int.J.Mod.Phys. **A25** 332 (2010) [arXiv:0909.1580[hep-th]].
- [9] D.Martelli and J. Sparks, “Moduli spaces of Chern-Simons quiver gauge theories and AdS₄/CFT₃,” Phys.Rev. **D78**, 126005 (2008) [arXiv:0808.0912[hep-th]].
- [10] B.Feng, A.Hanany and Y.H.He, “D-brane gauge theories from toric singularities and toric duality,” Nucl. Phys. **B595**, 165 (2001) [arXiv:hep-th/0003085].
- [11] A. Hanany and K.D. Kennaway, “Dimer models and toric diagrams,” [arXiv:hep-th/0503149].
- [12] S.Franco, A.Hanany, K.D. Kennaway, D.Vegh and B.Weht, “Brane dimers and quiver gauge theories,” JHEP **0601**, 096 (2006) [arXiv:hep-th/0504110].
- [13] A.Hanany and A.Zaffaroni, “Tilings, Chern-Simons Theories and M2 branes,” JHEP **0810**, 111 (2008) [arXiv:0808.1244[hep-th]].
- [14] K.Ueda and M.Yamazaki, “Toric Calabi-Yau four-folds dual to Chern-Simons-matter theories,” JHEP **0812**, 045 (2008) [arXiv:0808.3768[hep-th]].
- [15] A.Hanany, D.Vegh, A.Zaffaroni, “Brane Tilings and M2 branes,” JHEP **0903**, 012 (2009) [arXiv:0809.1440].
- [16] S.Franco, A.Hanany, J.Park and D.Rodriguez-Gomez, “Towards M2-brane Theories for Generic Toric Singularities,” JHEP **0812**, 110 (2008) [arXiv:0809.3237 [hep-th]].
- [17] A.Hanany and Y.H.He, “M2-branes and Quiver Chern-Simons: A Taxonomic Study,” arXiv:0811.4044[hep-th].
- [18] J.Davey, A.Hanany, N.Mekareeya and G.Torri, “Phases of M2-brane Theories,” JHEP **0906**, 025 (2009) [arXiv:0903.3234[hep-th]].

- [19] J.Davey, A.Hanany, N.Mekareeya and G.Torri, “Higgsing M2-brane Theories,” arXiv:0908.4033[hep-th].
- [20] J.Davey, A.Hanany, N.Mekareeya and G.Torri, “M2-Branes and Fano 3-folds,” arXiv:1103.0553[hep-th].
- [21] K. Watanabe, M. Watanabe, Tokyo J. Math, 5, no. 1 (1982).
- [22] S. Dwivedi, P. Ramadevi, “Inverse algorithm and M2-brane theories,” JHEP **1111**, 111 (2011) [arXiv:1108.2387v3 [hep-th]].
- [23] P. Agarwal, P. Ramadevi, T. Sarkar, “A note on dimer models and D-brane gauge theories,” JHEP **0806** 054 (2008) [arXiv:0804.1902].
- [24] Prabwal Phukon, Tapobrata Sarkar, “On the Higgsing and UnHiggsing of Fano 3-Folds,” JHEP **1201** 090 (2012) [arXiv:1108.4237v1].